viii) De-Morgan Law :- B be Boolean Algebra.

$a, b \in B$

T.P i) $(a+b)' = a' \cdot b'$

ii) $(a \cdot b)' = a' + b'$

Proof i) T.P Complement of $a+b$ is $a' \cdot b'$

$\therefore$ i.e. $(a+b) + (a' \cdot b') = 1$

& $(a+b) \cdot (a' \cdot b') = 0$

| $a + \bar{a} = 1$
| $a \cdot \bar{a} = 0$
| $a + 1 = 1$
| $a \cdot 0 = 0$

Now L.H.S $(a+b) + (a' \cdot b')$

$= [(a+b) + a'] \cdot [(a+b) + b']$ (By distr.)

$= [(b+a) + a'] \cdot [a + (b+b')]$ (By Comm.)

$= (b+1) \cdot (a+1)$ (by Compli.)

$= 1 \cdot 1$ (By Add Law)

$= 1$ = R.H.S.

also L.H.S $(a+b) \cdot (a' \cdot b')$

$= \cancel{(a' + b') \cdot (a+b)}$ ~~(By Commut.)~~

$\cancel{= (a' \cdot a)}$

$= a \cdot (a' \cdot b') + b \cdot (a' \cdot b')$ (By dist.)

$= (a \cdot a') \cdot b' + b \cdot (b' \cdot a')$ (By Commut.)

$= (0 \cdot b') + (b \cdot b') \cdot a'$ (By Compli.)

$= 0 + 0 \cdot a'$

$= 0 + 0 = 0 =$ R.H.S

ii) T.P Complement of $a \cdot b$ is $a' + b'$

i.e. $(a \cdot b) + (a' + b') = 1$

& $(a \cdot b) \cdot (a' + b') = 0$

L.H.S $(a \cdot b) + (a' + b')$

$= [a + (a' + b')] \cdot [b + (a' + b')]$

$= [(a + a') + b'] + [(b + b') + a']$ (By dist. & Comm.)

$= (1 + b') + (1 + a')$ (By Comple.)

$= 1 \cdot 1$

$= 1 = R.H.S.$

L.H.S $(a \cdot b) \cdot (a' + b') \Rightarrow (a \cdot b) \cdot a' + (a \cdot b) \cdot b'$ (By dist.)

$\Rightarrow (b \cdot a) \cdot a' + a \cdot (b \cdot b')$ (By Comm.)

$\Rightarrow b \cdot (a \cdot a') + a \cdot 0$ (By Comple.)

$\Rightarrow b \cdot 0 + a \cdot 0$

$\Rightarrow 0 + 0$

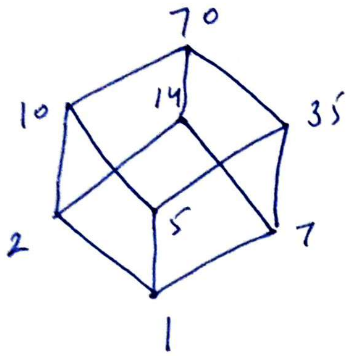
$\Rightarrow 0 = R.H.S$

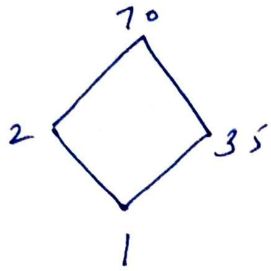
Hence Proved.

4. Boolean Sub-Algebra :- Let $\langle B, \vee, \wedge, ', 0, 1 \rangle$ be a Boolean algebra and $S \subseteq B$ contain elements of 0 & 1 and is closed under the operations $\vee, \wedge, '$ then $\langle S, \vee, \wedge, ', 0, 1 \rangle$ is called B.S.A. [l.u.b of S = l.u.b of B & g.l.b of S = g.l.b of B]

eg. $D_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$ is Boolean Algebra & $\{1, 2, 35, 70\} \subseteq D_{70}$ T.P $\{1, 2, 35, 70\}$ is Bool. Sub Algebra.

Sol.





Let $S = \{1, 2, 35, 70\}$

$a, b \in S$ then $a \vee b \in S$

$a \wedge b \in S$

$a', b' \in S$

least Element of B and S = 1

Greatest Element of B and S = 70.

i.e. l.u.b of B = 70 = l.u.b of S

g.l.b. of B = 1 = g.l.b of S

$\therefore$ S is Boolean Sub-Algebra.

5. Direct Product of Boolean Algebra :-

Let $\langle B_1, \vee_1, \wedge_1, ', 0_1, 1_1 \rangle$ & $\langle B_2, \vee_2, \wedge_2, '', 0_2, 1_2 \rangle$ be two Boolean Algebra.

Scanned with ACE Scanner

Direct Product = $< B_1 \times B_2 , \wedge_3, \vee_3, ''', O_3, 1_3 >$

s.t $(a_1, b_1), (a_2, b_2) \in B_1 \times B_2$

as $(a_1, b_1) \wedge_3 (a_2, b_2) = (a_1 \wedge_1 a_2 , b_1 \wedge_2 b_2)$

$(a_1, b_1) \vee_3 (a_2, b_2) = (a_1 \vee_1 a_2 , b_1 \vee_2 b_2)$

$(a_1, b_1)''' = (a_1' , b_1'')$

$O_3 = (O_1, O_2)$

$1_3 = (1_1, 1_2)$

Thm Direct Product of two Boolean Algebras is a Boolean Algebra.

Sol: Let A & B be two Boolean Algebra.
(B_1) (B_2)

$\therefore$ B_1 & B_2 are bdd, Complemented distributive lattice.

T.P $B_1 \times B_2$ is a Boolean Algebra.
ie. bdd, Comple. distr. lattice.

we know direct Product of two lattice is a lattice

$\therefore$ $B_1 \times B_2$ is a lattice.

I Bounded :- Given B_1 & B_2 are bdd.

let O_1, O_2 least Elements.

& $1_1, 1_2$ Greatest Elements of B_1 & B_2.

Then $(0_1, 0_2)$ and $(1_1, 1_2)$ are least & Greatest Elements of $B_1 \times B_2$.

$\therefore$ $B_1 \times B_2$ is bounded.

II <u>Complemented</u> :- Let $(a, b) \in B_1 \times B_2$.

$\Rightarrow$ $a \in B_1$ and $b \in B_2$

$\exists$ $a' \in B_1$ and $b' \in B_2$

$a \vee a' = 1_1$, $a \wedge a' = 0_1$.

& $b \vee b' = 1_2$, $b \wedge b' = 0_2$

Also $(a', b') \in B_1 \times B_2$

Now $(a, b) \vee (a', b') = (a \vee a', b \vee b')$

$= (1_1, 1_2)$

& $(a, b) \wedge (a', b') = (a \wedge a', b \wedge b')$

$= (0_1, 0_2)$

$\Rightarrow$ (a', b') is Complemented of (a, b) in $B_1 \times B_2$.

Hence $B_1 \times B_2$ is Complemented lattice.

III <u>Distributive</u> :- Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in B_1 \times B_2$ three Elements

$\therefore$ $a_1, a_2, a_3 \in B_1$ & $b_1, b_2, b_3 \in B_2$

Given B_1 & B_2 are distributive lattice.

Scanned with ACE Scanner

$\therefore\ a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$ — (1)

& $b_1 \wedge (b_2 \vee b_3) = (b_1 \wedge b_2) \vee (b_1 \wedge b_3)$ — (2)

Now $(a_1, b_1) \wedge [(a_2, b_2) \vee (a_3, b_3)]$

L.H.S

$\Rightarrow (a_1, b_1) \wedge (a_2 \vee a_3, b_2 \vee b_3)$

$\Rightarrow [(a_1 \wedge a_2) \vee (a_1 \wedge a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)]$ — (3)

R.H.S. $[(a_1, b_1) \wedge (a_2, b_2)] \vee [(a_1, b_1) \wedge (a_3, b_3)]$

$\Rightarrow (a_1 \wedge a_2, b_1 \wedge b_2) \vee (a_1 \wedge a_3, b_1 \wedge b_3)$

$\Rightarrow [(a_1 \wedge a_2) \vee (a_1 \wedge a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)]$ — (4)

$\therefore$ L.H.S = R.H.S

Hence Distributive Law holds in $B_1 \times B_2$

$\therefore$ $B_1 \times B_2$ is a Boolean Algebra.

6. Boolean Algebra Isomorphism:-

Let B_1 & B_2 be two Boolean algebras.

$f: B_1 \to B_2$ is called B.A.I

if i) f is 1-1

ii) f is onto

iii) $f(a+b) = f(a) + f(b)$

$f(ab) = f(a) \cdot f(b)$

iv) $f(a') = [f(a)]' \ \forall a \in B$ $\therefore B_1 \cong B_2$

Scanned with ACE Scanner